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Three-loop beta function and non-perturbative α_s in asymmetric momentum scheme

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Abstract: We determine the three-loop coefficient of the beta function in the asymmetric momentum subtraction scheme in Landau gauge. This scheme is convenient for lattice studies of α_s, the running coupling constant of QCD. We present high statistics lattice results for α_s in the SU(3) Yang-Mills theory without quark, compare with the three-loop running and extract the value of the corresponding $\Lambda_{\overline{\rm MS}}$ parameter. We estimate the systematic error coming from four-loop terms. We obtain the result: $\Lambda_{\overline{\rm MS}} = 295\,(5)(15)\frac{a^{-1}(\beta=6.0)}{1.97\text{GeV}}$ MeV.

Keywords: Lattice Gauge Field Theories, Lattice QC.

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Contents

Momentum subtraction schemes are interesting because they provide regularization independent renormalization methods which can be used non-perturbatively, contrary to the popular $\overline{\mathrm{MS}}$ schemes based on the dimensional regularization. Realized on the lattice, momentum subtraction schemes eliminate the need for lattice perturbation theory which has proved to be complicated and not very accurate. Usefulness of these schemes is exemplified in the non-perturbative renormalization of operators [1] which becomes widely used by now in the lattice approach.

In this paper we determine the three-loop running in an asymmetric momentum subtraction scheme, compare with our high statistics numerical results for α_s and extract the value for the $\Lambda_{\overline{\mathrm{MS}}}$ parameter in the SU(3) Yang-Mills theory without quark. In section 1, we recall the definitions of the $\widetilde{\mathrm{MOM}}$ renormalization scheme and define $g_{\widetilde{\mathrm{MOM}}}$, a coupling constant convenient for lattice studies [2, 3]. In section 2, we give the relation between this coupling constant and some quantities computed in the $\overline{\mathrm{MS}}$ scheme in ref. [4]. In section 3, we extract the three-loop coefficient of the beta function in the $\widetilde{\mathrm{MOM}}$ scheme and in section 4, we present our results for the coupling constant and the $\Lambda_{\overline{\mathrm{MS}}}$ parameter.

1. Definition of the coupling constant in the $\widetilde{\mathrm{MOM}}$ scheme

In this section, we suppose that we are able to compute or measure in some way the two- and three-point gluonic Green functions and define a convenient coupling constant.

When one of the momenta is equal to zero, the Lorentz structure of the three-point Green function, $G^{(3)abc}_{\mu\nu\rho}(p,-p,0)$, can depend a priori on four tensors : $\delta_{\mu\nu}p_\rho$,

$\delta_{\mu\rho}p_\nu$, $\delta_{\nu\rho}p_\mu$ and $p_\mu p_\nu p_\rho$. In Landau gauge, only one possibility is left by the transversality conditions and we can write :

$$G^{(3)abc}_{\mu\nu\rho}(p,-p,0) \;=\; 2\, f^{abc}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) p_\rho\, G^{(3)}(p^2) \tag{1.1}$$

for the bare Green function. Note that only the completely antisymmetric group structure functions f^{abc} can appear in this expression : the Bose symmetry of the three point function, i.e. the symmetry over $p \leftrightarrow -p$, $\mu \leftrightarrow \nu$ and $a \leftrightarrow b$, combined with the oddness of the Lorentz tensor when $p \leftrightarrow -p$ forbids terms with the symmetric color structure d^{abc}.

The bare two-point Green function in Landau gauge writes :

$$\begin{aligned} G^{(2)ab}_{\mu\nu}(p,-p) &= \delta^{ab}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\, G^{(2)}(p^2)\,, \\ G^{(2)ab}_{\mu\nu}(0,0) &= \delta^{ab}\delta_{\mu\nu}\, G^{(2)}(0)\,. \end{aligned} \tag{1.2}$$

The scalars functions $G^{(3)}$ and $G^{(2)}$ can be extracted from the Green functions with :

$$\begin{aligned} G^{(3)}(p^2) &= \frac{1}{144\, p^2}\, f^{abc}\delta^{\mu\nu}\, p^\rho\, G^{(3)abc}_{\mu\nu\rho}(p,-p,0)\,, \\ G^{(2)}(p^2) &= \frac{1}{24}\sum_{a,\mu} G^{(2)aa}_{\mu\mu}(p,-p)\,, \\ G^{(2)}(0) &= \frac{1}{32}\sum_{a,\mu} G^{(2)aa}_{\mu\mu}(0,0)\,. \end{aligned} \tag{1.3}$$

The wave function renormalization for the gluon, $Z_{\widetilde{\mathrm{MOM}}}$, and the renormalized coupling constant in the $\widetilde{\mathrm{MOM}}$ scheme are then defined by:

$$Z_{\widetilde{\mathrm{MOM}}}(\mu^2) = p^2\; G^{(2)}(p^2)\big|_{p^2=\mu^2}\,, \tag{1.4}$$

$$g_{\widetilde{\mathrm{MOM}}}(\mu^2) = \left.\frac{G^{(3)}(p^2)\; Z^{3/2}_{\widetilde{\mathrm{MOM}}}(\mu^2)}{G^{(2)}(p^2)G^{(2)}(p^2)G^{(2)}(0)}\right|_{p^2=\mu^2}\,, \tag{1.5}$$

where μ is the renormalization scale. The interpretation of eqs. (1.4-1.5) is standard: the momentum scheme fixes the renormalization constants so that the two- and three-point functions take their tree values with the substitution of the bare coupling by the renormalized one. In the usual MOM scheme, one chooses to work at the symmetric Euclidean point $p_1^2 = p_2^2 = p_3^2 = \mu^2$. Here the $\widetilde{\mathrm{MOM}}$ scheme is defined from an asymmetric point, when one of the momenta is equal to zero.

2. Connection with $\overline{\text{MS}}$ calculations

In ref. [4], the two- and three-point Green functions have been computed at two-loop with the dimensional regularization in an arbitrary covariant gauge. They define the (bare) form factors from the (bare) vertex function:

$$\Gamma^{abc}_{\mu_1\mu_2\mu_3}(p,-p,0) = f^{abc}\Big\{(2\delta_{\mu_1\mu_2}p_{\mu_3} - \delta_{\mu_1\mu_3}p_{\mu_2} - \delta_{\mu_2\mu_3}p_{\mu_1})T_1(p^2) \\ - p_{\mu_3}\left(\delta_{\mu_1\mu_2} - \frac{p_{\mu_1}p_{\mu_2}}{p^2}\right)T_2(p^2)\Big\}\,. \tag{2.1}$$

At zeroth order in perturbation theory, $T_1 = 1$ and $T_2 = 0$.

If we restrict to Landau gauge and insert these form factors into the definition (1.5) for the coupling constant in the $\widetilde{\text{MOM}}$ scheme, we get :

$$g_{\widetilde{\text{MOM}}}(\mu^2) = \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^{-\epsilon/2} g_0\left(T_1 - \frac{1}{2}T_2\right)\Big|_{p^2=\mu^2} Z^{3/2}_{\widetilde{\text{MOM}}}\,, \tag{2.2}$$

where g_0 is the bare coupling constant and a common scale, μ^2, has been chosen for the renormalization point used to define $g_{\widetilde{\text{MOM}}}$ and for the dimensional regularization.

To renormalize in the $\overline{\text{MS}}$ scheme, renormalization constants and a renormalized coupling constant have to be introduced. The definitions are:

$$\Gamma^{\mu_1\mu_2\mu_3}_{\overline{\text{MS}}}(p,-p,0) = Z_1\ \Gamma^{\mu_1\mu_2\mu_3}(p,-p,0)\,, \tag{2.3}$$

$$G^{(2)}_{\overline{\text{MS}}}(p^2) = Z_3^{-1}G^{(2)}(p^2)\,, \tag{2.4}$$

$$g_0 = \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^{\epsilon/2} g_{\overline{\text{MS}}}\ Z_1 Z_3^{-3/2}\,, \tag{2.5}$$

where the suffix $\overline{\text{MS}}$ indicates renormalized quantities in the $\overline{\text{MS}}$ scheme. Z_1 and Z_3 are the renormalization constants in the $\overline{\text{MS}}$ scheme; they are defined in the usual way to remove the $1/\epsilon^j$ singular terms present in the bare quantities. From (2.3) we get $T_{1\overline{\text{MS}}} = Z_1 T_1$ and $T_{2\overline{\text{MS}}} = Z_1 T_2$ and from the definitions (1.4) and (2.4):

$$p^2 G^{(2)}_{\overline{\text{MS}}}(p^2)\Big|_{p^2=\mu^2} = Z_3^{-1}p^2G^{(2)}(p^2)\Big|_{p^2=\mu^2} = \frac{Z_{\widetilde{\text{MOM}}}}{Z_3}\,. \tag{2.6}$$

Collecting everything, the relation between the $\widetilde{\text{MOM}}$ coupling constant and $\overline{\text{MS}}$ quantities is:

$$\alpha_{\widetilde{\text{MOM}}}(\mu^2) = \alpha_{\overline{\text{MS}}}(\mu^2)\left(T_{1\overline{\text{MS}}} - \frac{1}{2}T_{2\overline{\text{MS}}}\right)^2\Big|_{p^2=\mu^2}\left(\mu^2 G^{(2)}_{\overline{\text{MS}}}(\mu^2)\right)^3\,, \tag{2.7}$$

where $\alpha_{\widetilde{\text{MOM}}} \equiv g^2_{\widetilde{\text{MOM}}}/4\pi$ and $\alpha_{\overline{\text{MS}}} \equiv g^2_{\overline{\text{MS}}}/4\pi$.

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3. The three-loop β function in the $\widetilde{\mathrm{MOM}}$ scheme

The β function in the $\widetilde{\mathrm{MOM}}$ and $\overline{\mathrm{MS}}$ schemes are defined by:

$$\beta_{\widetilde{\mathrm{MOM}}}(\alpha_{\widetilde{\mathrm{MOM}}}) = \mu\frac{\partial\alpha_{\widetilde{\mathrm{MOM}}}}{\partial\mu} = -\frac{\tilde\beta_0}{2\pi}\alpha^2_{\widetilde{\mathrm{MOM}}} - \frac{\tilde\beta_1}{4\pi^2}\alpha^3_{\widetilde{\mathrm{MOM}}} - \frac{\tilde\beta_2}{64\pi^3}\alpha^4_{\widetilde{\mathrm{MOM}}} - \frac{\tilde\beta_3}{128\pi^4}\alpha^5_{\widetilde{\mathrm{MOM}}} - \cdots$$

$$\beta_{\overline{\mathrm{MS}}}(\alpha_{\overline{\mathrm{MS}}}) = \mu\frac{\partial\alpha_{\overline{\mathrm{MS}}}}{\partial\mu} = -\frac{\beta_0}{2\pi}\alpha^2_{\overline{\mathrm{MS}}} - \frac{\beta_1}{4\pi^2}\alpha^3_{\overline{\mathrm{MS}}} - \frac{\beta_2}{64\pi^3}\alpha^4_{\overline{\mathrm{MS}}} - \cdots \tag{3.1}$$

From [4], the perturbative expansions for the gluon propagator and the form factors $T_{1\overline{\mathrm{MS}}}$ and $T_{2\overline{\mathrm{MS}}}$ are known up to two loops in any covariant gauge. Eq. (2.7) can then be written as:

$$\alpha_{\widetilde{\mathrm{MOM}}} = \alpha_{\overline{\mathrm{MS}}}\left(1 + a\,\frac{\alpha_{\overline{\mathrm{MS}}}}{4\pi} + b\,\frac{\alpha^2_{\overline{\mathrm{MS}}}}{16\pi^2} + \cdots\right), \tag{3.2}$$

with known coefficient a and b. This gives a relation between the β functions:

$$\beta_{\widetilde{\mathrm{MOM}}}(\alpha_{\widetilde{\mathrm{MOM}}}) = \beta_{\overline{\mathrm{MS}}}(\alpha_{\overline{\mathrm{MS}}})\left(1 + a\,\frac{\alpha_{\overline{\mathrm{MS}}}}{2\pi} + 3b\,\frac{\alpha^2_{\overline{\mathrm{MS}}}}{16\pi^2} + \cdots\right). \tag{3.3}$$

Using eq. (3.2) and the definitions (3.1), we can expand (3.3) in power of $\alpha_{\overline{\mathrm{MS}}}$ and identify the terms with the same power; we get:

$$\tilde\beta_0 = \beta_0\,, \tag{3.4}$$

$$\tilde\beta_1 = \beta_1\,, \tag{3.5}$$

$$\tilde\beta_2 = \beta_2 + 2\left(b - a^2\right)\beta_0 - 4a\beta_1\,. \tag{3.6}$$

Eqs. (3.4) and (3.5) are usual results: the two first coefficients of the β function do not depend on the renormalization scheme. Eq. (3.6) tells us that the two-loop results in $\overline{\mathrm{MS}}$ for the gluon propagator, T_1 and T_2 (available in [4]) are sufficient to get the difference between the three-loop coefficients of the β function in the $\widetilde{\mathrm{MOM}}$ and the $\overline{\mathrm{MS}}$ schemes. In eq. (3.6) a, b and β_2 have to be computed in Landau gauge. But it is known that β_2 in $\overline{\mathrm{MS}}$ does not depend on the gauge [5] so we can use the standard result in Feynman gauge, $\beta_2 = 2857 - \frac{5033}{9}\,N_f + \frac{325}{27}\,N_f^2$ [6].

Finally, the three-loop β function in the $\widetilde{\mathrm{MOM}}$ scheme and Landau gauge is found to be:

$$\tilde\beta_0 = 11 - \frac{2}{3}N_f\,, \qquad \tilde\beta_1 = 51 - \frac{19}{3}N_f\,,$$

$$\tilde\beta_2 = \frac{186747}{32} - \frac{1683}{2}\zeta_3 - \left(\frac{35473}{48} - \frac{65}{3}\zeta_3\right)N_f - \\ - \left(\frac{829}{27} - \frac{16}{9}\zeta_3\right)N_f^2 + \frac{16}{9}\,N_f^3\,, \tag{3.7}$$

where $\zeta_3 \simeq 1.2020569\ldots$ is the value of the Riemann's zeta function and N_f is the number of flavor. In the following we will work in the flavorless case, $N_f = 0$, for which $\tilde\beta_2 \simeq 4824.31$.

4. $\Lambda_{\overline{\rm MS}}$ from $\alpha_{\widetilde{\rm MOM}}$

The implementation of the $\widetilde{\rm MOM}$ scheme on the lattice is straightforward [2, 3]. Varying the external momentum, the $\widetilde{\rm MOM}$ coupling constant can be obtained directly in one simulation for several values of the scale μ^2. In ref. [3], the two- and three-point Green functions have been measured on the lattice in Landau gauge with high statistics and for several lattice spacings and volumes in the flavorless case. For large values of the momentum, lattice artifacts of $O(a^2p^2)$ affect the Green functions. Numerically as expected these effects are seen to decrease when β increases. On the lattice the gauge fixing algorithm leads to the relation $\frac{2}{a}\sin(\frac{ap_\mu}{2})A_\mu(p) = 0$ while $p_\mu A_\mu(p)$ does not vanish. And actually we have seen on our data [3] that the dominant part of $O(a^2p^2)$ artifacts in $\alpha_{\widetilde{\rm MOM}}$ is corrected by the substitution of the momenta p_μ by the lattice momenta $\frac{2}{a}\sin(\frac{ap_\mu}{2})$ in eqs. (1.1) and (1.2). This was already noticed in [2]; in other contexts, authors have shown that this choice for the lattice momentum was indeed favored by their data, see for example [9, 7].

We give now some results from the three-loop analysis, see [3] for details on the lattice settings. In fig. 1, we give the behavior of the coupling constant, $\alpha_{\widetilde{\rm MOM}}$, as a function of the scale μ obtained from the simulation and compared with the integration of the three-loop beta-function. A nice scaling is apparent for scales larger than 2 GeV.

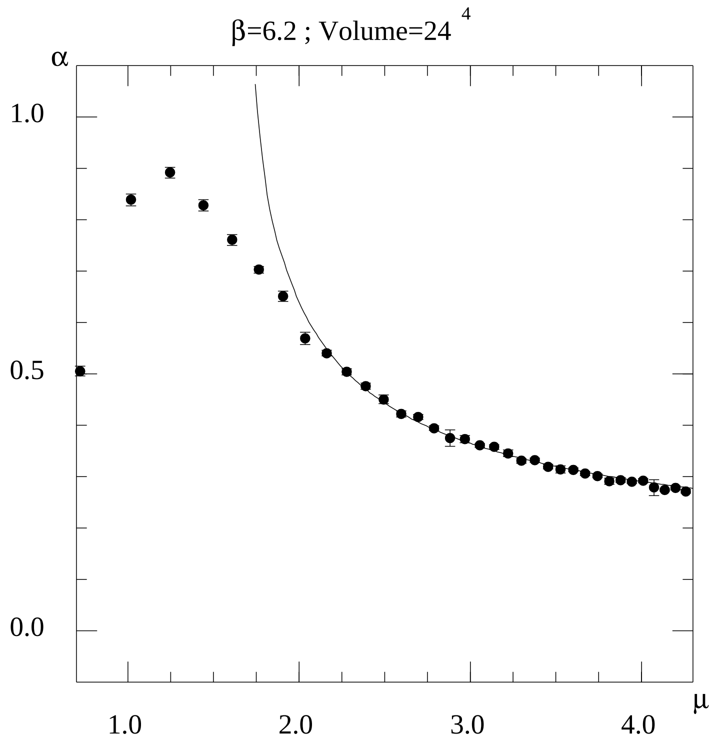

Figure 1: The QCD coupling constant $\alpha_{\widetilde{\rm MOM}}$ as a function of the scale μ (in GeV). The full line is the three-loop running.

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To calibrate the lattice at $\beta = 6.2$, we have used the value for the the lattice spacing which has been measured recently with a non-perturbatively improved action (free from O(a) artifacts) [7]. The measured value is: $a^{-1}(\beta = 6.2) = 2.75(18)$ GeV. Other lattices have been calibrated relatively to the one at $\beta = 6.2$ with the results for $a\sqrt{\sigma}$, the string tension in lattice units, published in [8]. We took: $a^{-1}(\beta = 6.0) = 1.97$ GeV, $a^{-1}(\beta = 6.2) = 2.75$ GeV and $a^{-1}(\beta = 6.4) = 3.66$ GeV.

Another way to exhibit the scaling is to extract a Λ parameter as a function of the scale and look for a plateau at high scale. Like in [10] we define in general the Λ parameter as:

$$\begin{aligned}\Lambda \equiv \mu \left(\frac{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}{4\pi}\right)^{-\frac{\tilde\beta_1}{\tilde\beta_0^2}} \exp\left\{\frac{-2\pi}{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}\right\} \times \\ \times \exp\left\{-\int_0^{\alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)} d\alpha \left[\frac{1}{\beta(\alpha)} + \frac{2\pi}{\tilde\beta_0 \alpha^2} - \frac{\tilde\beta_1}{\tilde\beta_0^2 \alpha}\right]\right\}.\end{aligned} \tag{4.1}$$

If we consider the expansion of the $\widetilde{\mathrm{MOM}}$ β function truncated at three-loop, (4.1) can be integrated to give:

$$\begin{aligned}\Lambda_{\widetilde{\mathrm{MOM}}} = \mu \left(\frac{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}{4\pi}\right)^{-\frac{\tilde\beta_1}{\tilde\beta_0^2}} \exp\left\{\frac{-2\pi}{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}\right\} \times \left(1 + \frac{\tilde\beta_1\ \alpha_{\widetilde{\mathrm{MOM}}}}{2\pi\tilde\beta_0} + \frac{\tilde\beta_2\ \alpha^2_{\widetilde{\mathrm{MOM}}}}{32\pi^2\tilde\beta_0}\right)^{\frac{\tilde\beta_1}{2\tilde\beta_0^2}} \times \\ \times \exp\left\{\frac{\tilde\beta_0\tilde\beta_2 - 4\tilde\beta_1^2}{2\tilde\beta_0^2\sqrt{\Delta}} \left[\arctan\left(\frac{\sqrt{\Delta}}{2\tilde\beta_1 + \tilde\beta_2 \alpha_{\widetilde{\mathrm{MOM}}}/4\pi}\right) - \arctan\left(\frac{\sqrt{\Delta}}{2\tilde\beta_1}\right)\right]\right\},\end{aligned} \tag{4.2}$$

where $\Delta \equiv 2\tilde\beta_0\tilde\beta_2 - 4\tilde\beta_1^2$ ($\Delta > 0$ in our case). This can be consistently expanded to give:

$$\Lambda_{\widetilde{\mathrm{MOM}}} \simeq \mu \left(\frac{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}{4\pi}\right)^{-\frac{\tilde\beta_1}{\tilde\beta_0^2}} \exp\left\{\frac{-2\pi}{\tilde\beta_0 \alpha_{\widetilde{\mathrm{MOM}}}(\mu^2)}\right\} \times \left(1 + \frac{8\tilde\beta_1^2 - \tilde\beta_0\tilde\beta_2}{16\pi\tilde\beta_0^3}\alpha_{\widetilde{\mathrm{MOM}}}\right). \tag{4.3}$$

For each value of the scale μ, there is a corresponding $\alpha_{\widetilde{\mathrm{MOM}}}$ and we can associate an effective $\Lambda_{\widetilde{\mathrm{MOM}}}$ parameter through the formulas above. This effective $\Lambda_{\widetilde{\mathrm{MOM}}}$ parameter should become a constant at sufficiently high μ^2 when the scaling settles. It should be noted that (4.2) gives larger plateaux than (4.3). But independently of the formula used to extract Λ, an estimate for the systematic error due to the influence of higher-order terms on this determination is needed. It can be obtained if we add by hand a four-loop term in the beta function and vary its coefficient $\tilde\beta_3$ over a reasonable interval. Evaluation of these effects will be given below.

The asymptotic ratio $\frac{\Lambda_{\widetilde{\mathrm{MOM}}}}{\Lambda_{\overline{\mathrm{MS}}}}$ is known exactly [2]: $\Lambda_{\overline{\mathrm{MS}}} = \exp\{-\frac{70}{66}\} \times \Lambda_{\widetilde{\mathrm{MOM}}} \simeq 0.346 \times \Lambda_{\widetilde{\mathrm{MOM}}}$. Let us forget for the moment the influence of higher order terms and use the three-loop expression.

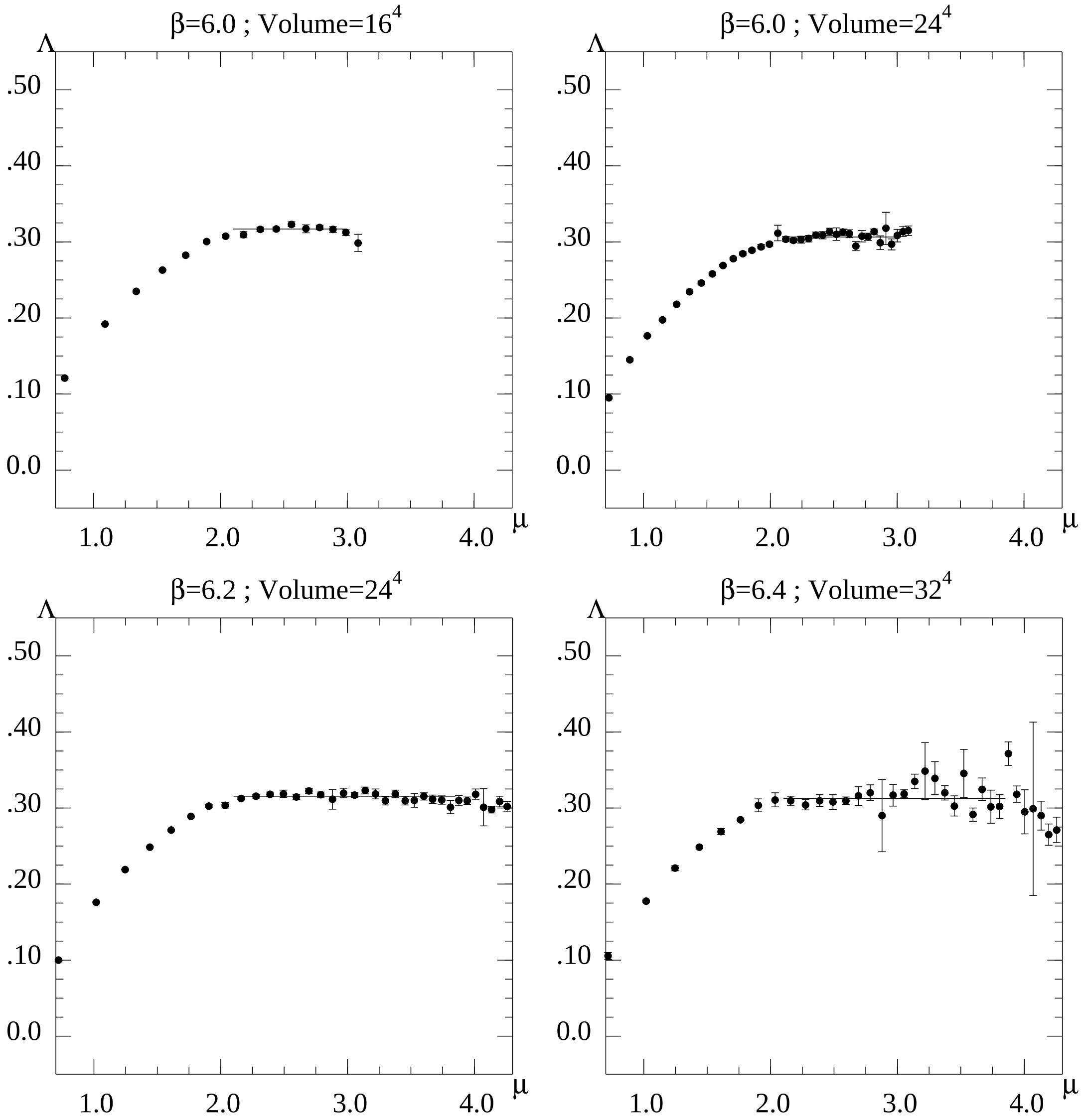

Figure 2: Λ_{QCD} (in GeV) as a function of the scale μ (in GeV) for different lattice spacings and volumes.

In fig. 2 we plot the effective $\Lambda_{QCD} \equiv 0.346 \times \Lambda_{\widetilde{\text{MOM}}}$ as a function of μ for several values of the volume and the lattice spacing. Scaling manifests itself through the plateaux of Λ_{QCD} at large enough scale. The value of the plateau gives our "measurement" for the flavorless $\Lambda_{\overline{\text{MS}}}$. From our largest physical volume $(\beta, V) = (6.0, 24^4)$ (for which $V_{\text{phys}}^{-1/4} \simeq 2.4$ fm) we obtain: $\Lambda_{\overline{\text{MS}}} = (303 \pm 5\,\text{MeV})$. For comparison we give the results from the other lattices at $(\beta = 6.0, V = 16^4)$, $(\beta = 6.2, V = 24^4)$ and $(\beta = 6.4, V = 32^4)$, all three with nearly the same physical volume, $V_{\text{phys}}^{1/4} \simeq 1.7$ fm. We found $\Lambda_{\overline{\text{MS}}} = 314(3)$, $313(4)$ and $312(9)$ respectively. Scaling in β is striking and a comparison between the two different physical volumes shows that finite volume effects are moderate (a few %).

As explained previously, to estimate the systematic error from unknown higher order terms in the perturbative expansion we add a four-loop term in the beta-function and study the variation of Λ as a function of the unknown coefficient $\tilde{\beta}_3$ varying up to some $\tilde{\beta}_3^{lim}$. We conventionally choose to limit this interval when the four-loop term in the beta function is equal in magnitude to the three-loop one at $\alpha = \alpha^{lim} \equiv 0.4$, namely $\tilde{\beta}_3^{lim} = \frac{2\pi}{\alpha^{lim}}\tilde{\beta}_2$. We analyze the data and extract the values for $\Lambda_{\widetilde{\rm MOM}}$ with

$$\Lambda_{\widetilde{\rm MOM}} \simeq \mu \left(\frac{\tilde{\beta}_0 \alpha_{\widetilde{\rm MOM}}(\mu^2)}{4\pi}\right)^{-\frac{\tilde{\beta}_1}{\tilde{\beta}_0^2}} \exp\left\{\frac{-2\pi}{\tilde{\beta}_0 \alpha_{\widetilde{\rm MOM}}(\mu^2)}\right\} \times \left[1 + \frac{8\tilde{\beta}_1^2 - \tilde{\beta}_0\tilde{\beta}_2}{16\pi\tilde{\beta}_0^3}\alpha_{\widetilde{\rm MOM}} + \right.$$
$$\left. + \frac{\alpha^2_{\widetilde{\rm MOM}}}{2}\left(\frac{2\tilde{\beta}_0\tilde{\beta}_1\tilde{\beta}_2 - 8\tilde{\beta}_1^3 - \tilde{\beta}_0^2\tilde{\beta}_3}{32\pi^2\tilde{\beta}_0^4} + \left(\frac{8\tilde{\beta}_1^2 - \tilde{\beta}_0\tilde{\beta}_2}{16\pi\tilde{\beta}_0^3}\right)^2\right)\right]. \tag{4.4}$$

From the lattice with the largest physical volume this gives for our final result:

$$\Lambda_{\overline{\rm MS}} = 295\,(5)(15)\ \frac{a^{-1}(\beta = 6.0)}{1.97\text{GeV}}\ \text{MeV}\,. \tag{4.5}$$

where the first error comes from the statistics and the second one from the systematics attached with the higher order terms.

References

[1] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, *A general method for non-perturbative renormalization of lattice operators,* *Nucl. Phys.* **B 445** (1995) 81 [hep-lat/9411010].

[2] B. Alles, D. Henty, H. Panagopoulos, C. Parrinello, C. Pittori and D.G. Richards, *α_s from the nonperturbatively renormalized lattice three gluon vertex,* *Nucl. Phys.* **B 502** (1997) 325;
C. Parrinello, *Status of α_s determinations from the nonperturbatively renormalized three gluon vertex,* *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 245 [hep-lat/9710053];
B. Alles, D. Henty, H. Panagopoulos, C. Parrinello and C. Pittori, *α_s from the non-perturbatively renormalised lattice three-gluon vertex,* hep-lat/9605033.

[3] Ph. Boucaud, J.P. Leroy, J. Micheli, O. Pène and C. Roiesnel, *Lattice calculation of α_s in momentum scheme,* *JHEP* **10** (1998) 017 [hep-ph/9810322].

[4] A.I. Davydychev, P. Osland and O.V. Tarasov, *Two loop three gluon vertex in zero momentum limit,* *Phys. Rev.* **D 58** (1998) 036007 [hep-ph/9801380].

[5] S.A. Larin and J.A.M. Vermaseren, *The three loop QCD beta function and anomalous dimensions,* *Phys. Lett.* **B 303** (1993) 334 [hep-ph/9302208].

[6] O. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *The Gell-Mann-low function of QCD in the three loop approximation,* *Phys. Lett.* **B 93** (1980) 429.

[7] D. Becirevic, Ph. Boucaud, L. Giusti, J.P. Leroy, V. Lubicz, G. Martinelli, F. Mescia and F. Rapuano, *Light hadron spectroscopy on the lattice with the non-perturbatively improved Wilson action,* `hep-lat/9809129`.

[8] G.S. Bali and K. Schilling, *Running coupling and the lambda parameter from SU(3) lattice simulations,* *Phys. Rev.* **D 47** (1993) 661 [`hep-lat/9208028`].

[9] T. Battacharya, R. Gupta, G. Kilcup and S. Sharpe, *Hadron spectrum with Wilson fermions,* *Phys. Rev.* **D 53** (1996) 6486 [`hep-lat/9512021`];
C.R. Allton, V. Gimenez, L. Giusti and F. Rapuano, *Light quenched hadron spectrum and decay constants on different lattices,* *Nucl. Phys.* **B 489** (1997) 427 [`hep-lat/9611021`].

[10] M. Lüscher, *Advanced lattice QCD*, `hep-lat/9802029`.

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